working paper series

OPTIMAL ASSET ALLOCATION
AND CONSUMPTION RULES
FOR COMMODITY-BASED
SOVEREIGN WEALTH FUNDS

Khouzeima Moutanabbir and Diaa Noureldin

OPTIMAL ASSET ALLOCATION AND CONSUMPTION RULES FOR COMMODITY-BASED SOVEREIGN WEALTH FUNDS

Khouzeima Moutanabbir and Diaa Noureldin

Working Paper 1172

February 2018

Send correspondence to:
Diaa Noureldin
American University in Cairo and ERF
diaa.noureldin@aucegypt.edu

Optimal Asset Allocation and Consumption Rules for Commodity-Based Sovereign Wealth Funds

KHOUZEIMA MOUTANABBIR
Department of Mathematics and Actuarial Science
American University in Cairo
`k.moutanabbir@aucegypt.edu`

DIAA NOURELDIN
Department of Economics
American University in Cairo
`diaa.noureldin@aucegypt.edu`

February 3, 2018

Abstract

This paper studies the optimal asset allocation for a sovereign wealth fund (SWF) subject to a stochastic stream of commodity-based income, where, without loss of generality, we focus on oil-based SWFs. Using CRRA utility, we assume the fund's objective is to maximize the discounted utility of intertemporal consumption in the presence of time-varying investment opportunities, and given non-zero correlations between shocks to oil income and asset return innovations. We use the log-linear approximation method of Campbell (1993) to solve for the model's optimal asset allocation and consumption rules. Using historical data, we estimate the model parameters using the maximization-by-parts algorithm of Song et al. (2005). We then calibrate the model to study the optimal allocation and consumption for varying levels of risk aversion, time preference and oil income volatility. Our results are of interest to SWFs seeking optimal portfolio choice in the face of changing investment opportunities which correlate with their stochastic stream of income.

Keywords: Sovereign wealth fund; optimal asset allocation; optimal consumption rule, oil income volatility; hedging demand; CRRA utility.

JEL classification: C61; E21; G11; Q32.

1 Introduction

The proliferation of sovereign wealth funds (SWFs) in the last two decades is a notable evolution in how a country's national savings can be harnessed to achieve multiple objectives. For many states, their SWFs are perceived as vehicles for precautionary savings and intergenerational wealth transfer. In addition, they can and have been used for macroeconomic stabilization in the face of adverse economic shocks. The drawdowns witnessed in some commodity-dependent

SWFs in the face of the secular decline in commodity prices is a case in point, specifically for oil-based SWFs. Furthermore, some SWFs are thought to have been used to gain political leverage at opportune moments. Therefore, it is not surprising that they have attracted the attention of academics, practitioners and policymakers.

Perhaps their significance as key players in international financial markets came into the limelight during the 2008 financial crisis when some SWFs helped recapitalize some distressed European and U.S. banks. In 2009, and amid rising concerns about the potential use of SWFs for political objectives, the International Monetary Fund supported the establishment of the International Forum of Sovereign Wealth Funds, with the aim of encouraging best practices including transparency about the funds' objectives and operations.

The current total assets of SWFs are about US$ 7.4 trillion, of which 56.6 percent are in funds based on the proceeds from oil exports.[1] Since 2000, total assets have increased sevenfold in large part due to high oil prices, specifically during the period 2008-2013 save for the drop in 2009. Approximately 40 percent of the funds (by value) originated from the Middle East, with a similar proportion originating from Asia. Some of the world's largest SWFs comprise the bulk of accumulated national savings in their economies, and the fund value often exceeds aggregate output. For instance, with regard to the two largest SWFs in Norway and the UAE, their size was 2 to 3 times the GDP of their respective economies in 2016.

For the majority of SWFs, the allocation among asset classes has evolved considerably over the years, with the majority of the oil-based funds increasing their allocation to equity relative to bonds. For instance, the SWF of Norway, currently the largest in the world with US$ 922 billion in total assets, has gradually changed its equity-bond allocation from a 40-60 ratio in 2006 to a 60-40 ratio in 2009. The funds of Kuwait, Norway, Qatar, Saudi Arabia and UAE all have comparable allocations to equity, roughly equal to 60 percent of their portfolio allocation. For Norway and Saudi Arabia, about 35 percent of their SWF portfolio is invested in bonds, while Kuwait, Qatar and UAE invest 15-20 percent in bonds, and the rest is allocated to alternative investments such as real estate and direct equity in infrastructure projects.

The studies of Dyck and Morse (2011) and Bernstein et al. (2013) explored the determinants of the asset allocation of SWFs by studying their historical transactions and acquisitions. The primary conclusion of these studies is that the two most predominant objectives seem to be maximizing risk-adjusted returns, and fulfilling strategic objectives related to the development of

---

[1]This estimate is based on the Sovereign Wealth Fund Institute (SWFI) online database, available at https://www.swfinstitute.org. The estimate for most funds is for accumulated assets as of end March 2017.

know-how in particular industries. In contrast to the actual asset allocation for a SWF, this paper focuses on its optimal asset allocation given dependence on a stochastic stream of commodity-based income, which has a given correlation structure with the financial market variables.[2] In this regard, our paper contributes to the nascent but growing literature on optimal asset allocation for SWFs; see, for example, Gintschel and Scherer (2008), Scherer (2009), Balding and Yao (2011), Scherer (2011) and van den Bremer et al. (2016).

Gintschel and Scherer (2008), Balding and Yao (2011) and Scherer (2011) utilize the classical mean-variance framework of Markowitz (1959) and Lintner (1965a, 1965b), in which they deal with a static portfolio allocation problem with no intertemporal dimension. Our model is more in line with the class of dynamic asset allocation models pioneered by Merton (1969, 1971). In this class of models, the optimal asset allocation is intricately linked to the path of optimal consumption since the latter is the input to the investor's utility function. Our model also differs from the classical models in that it allows for a stochastic stream of income to influence the optimal asset allocation and the path of intertemporal consumption. In this regard, our model is closely related to the work of Veceira (2001) and Campbell et al. (2003). With regard to the literature on SWFs, our model's structure and assumptions also bear some resemblance to Scherer (2009) and van den Bremer et al. (2016). In what follows, we discuss how our model and assumptions differ from these earlier contributions.

Both Veceira (2001) and Campbell et al. (2003) build on a number of influential papers that focused on optimal asset allocation in the presence of a stochastic stream of labor income. Mayers (1972) and Fama and Schwert (1977) were among the first attempts to study the impact of human capital on portfolio choice, followed by the seminal works of Bodie et al. (1992) and Koo (1998), among others. We work under the assumption that the fund is used by the sovereign with the objective of smoothing intertemporal consumption out of the fund subject to stochastic returns on the fund's portfolio, and injections of new capital from a stochastic stream of oil revenue. Our setup is one where the fund is managed by an agent of the state with the sole objective of maximizing the utility of intertemporal consumption out of the fund, and we assume that the objective of the fund owner (i.e. the sovereign) coincides with that of the fund manager such that potential agency problems can be ignored.[3]

While Bodie et al. (1992) assumed that the exogenous stream of income is uncorrelated with

---

[2]Without loss of generality, we assume that the fund is based on oil revenues. Our analysis also extends straightforwardly to funds based on other commodities traded in international markets.

[3]See Bernstein et al. (2013) for a related discussion of the agency problems that arise from political interference with the fund management.

the returns on the risky assets, we allow the shocks to oil income and the innovations to the risky asset returns to be contemporaneously correlated, which gives room for hedging against oil income volatility. This also features in Gintschel and Scherer (2008), Scherer (2011) and van den Bremer et al. (2016). Given the objective of maximizing the utility of intertemporal consumption, our model also sheds light on the optimal consumption path under various assumptions about risk aversion, rate of time preference, and the correlation structure between oil income shocks and innovations to the risky assets and state variables. This, in turn, enables us to relate the optimal rules to the observed behavior of real-life SWFs.

Compared to Veceira (2001), our model allows for several risky assets in the portfolio choice problem. In addition, it allows for time-varying investment opportunities by specifying the dynamics of the excess returns on the risky assets and the state variables. This makes our model closer in structure to Campbell et al. (2003), however our model additionally includes a stochastic process for oil income, while having a stochastic stream of income is not considered in Campbell et al. (2003).

In the context of the literature on SWFs, our paper considers optimal asset allocation as an intertemporal problem as in Scherer (2009) and van den Bremer et al. (2016). Scherer (2009), however, does not derive an optimal rule for consumption out of the fund. On the other hand, van den Bremer et al. (2016) do not incorporate return predictability into their model. Our model includes both features, which enables us to study how the optimal allocation and path for consumption change with the available investment opportunities, and also to relate our findings to the actual behavior of existing oil-based SWFs. It is worth noting that we take the underground oil wealth as given, and thus our model does not include the optimal rate of oil extraction to manage the tradeoff between above- and underground wealth. This aspect of the problem is addressed in Scherer (2011) and van den Bremer et al. (2016) under different assumptions, and it is another point of distinction between their models and ours.

The rest of the paper is organized as follows: Section 2 gives an overview of SWFs with a particular focus on the largest oil-based SWFs. Section 3 presents the model and an outline of the solution for the optimal asset allocation and consumption rules, while Section 4 discusses the estimation of the model parameters. Section 5 presents the estimation results based on historical data, while Section 6 discusses the optimal asset allocation and consumption path, and shows how they change when varying the model's main parameters. Section 7 concludes the paper, while Appendix A includes the technical derivations.

2 Sovereign Wealth Funds: An Overview

With the exception of state-level funds in the U.S., such as the Permanent School Fund and Permanent University Fund at the State of Texas, the first national SWF (Kuwait Investment Authority) was established in Kuwait in 1953 with the objective of investing the surplus income from oil. Since then SWFs have proliferated and currently the largest funds in terms of asset value belong to Asian and the Middle Eastern states. As of end of March, 2017, SWFs assets stood at US$ 7.4 trillion, approximately 57 percent of which are in funds based on the proceeds of oil and gas exports. With the exception of China and Singapore, the world's largest SWFs belong to oil exporters, namely Kuwait, Qatar, Norway, Saudi Arabia and the UAE. The total assets of these funds increased significantly over the last decade due to a surge in oil export proceeds boosted by high oil prices.

Table 1 lists the largest SWFs around the globe, their total assets, year of inception and the origin of the fund. The world's largest SWF is the Government Pension Fund (Global) of Norway with assets amounting to US$ 922 billion as of end of March, 2017. For some countries, there exists more than one SWF. For instance, the UAE owns the Abu Dhabi Investment Authority (ADIA) with total assets of US$ 828 billion, in addition to four other SWFs with combined assets of US$ 479 billion. Each fund may have its own set of objectives. For example, the Abu Dhabi Investment Council, an offspring of ADIA, aims to maximize risk-adjusted returns through a well-diversified portfolio both locally and globally, whereas ADIA seeks long-term capital appreciation with partial hedging against oil price fluctuations.[4]

Oil-based SWFs enabled their states to accumulate a substantial wealth buffer during the prolonged boom in oil prices over the last decade, thereby creating a strong cushion for their respective economies as oil prices declined sharply in 2015. As seen in Table 2, the size of the SWFs in the UAE, Norway, Saudi Arabia, Kuwait and Qatar exceeded the level of annual output in 2016. In Kuwait and the UAE, the SWF assets were 4.6 and 3.2 times the size of the economy, respectively, while in Saudi Arabia, accumulated assets (in both SAMA and the Public Investment Fund) are almost on a par with GDP in 2016.

SWF assets are also a significant multiple of the annual oil rents[5] especially for the economies that are relatively more diversified such as Norway and the UAE.[6] The heterogeneity in the ratio

---

[4]Source: Funds' mission statements and objectives as stated on the Sovereign Wealth Fund Institute database.

[5]Oil rents are defined as the difference between the value of crude oil production at prevailing prices and the total cost of production, and is a measure popularized by the World Bank to capture the relative economic importance of extractive industries.

[6]It is worth noting that the figures for Qatar and Norway are likely understated since their energy wealth is

SWF Name	Country	Assets (US$ Bn.)	Inception	Origin
Government Pension Fund (Global)	Norway	922	1990	Oil
Abu Dhabi Investment Authority	UAE – Abu Dhabi	828	1976	Oil
China Investment Corporation	China	814	2007	Non-comm.
Kuwait Investment Authority	Kuwait	524	1953	Oil
SAMA Foreign Holdings[1]	Saudi Arabia	514	n.a.	Oil
Hong Kong Monetary Authority Investment Portfolio	China – Hong Kong	457	1993	Non-comm.
SAFE Investment Company[2]	China	441	1997	Non-comm.
Government of Singapore Investment Corporation	Singapore	359	1981	Non-comm.
Qatar Investment Authority	Qatar	320	2005	Oil & gas
National Social Security Fund	China	295	2000	Non-comm.
Investment Corporation of Dubai	UAE – Dubai	210	2006	Non-comm.
Temasek Holdings	Singapore	197	1974	Non-comm.
Public Investment Fund	Saudi Arabia	183	2008	Oil
Mubadala Investment Company	UAE – Abu Dhabi	125	2007	Oil
Abu Dhabi Investment Council	UAE – Abu Dhabi	110	2006	Non-comm.
Korea Investment Corporation	South Korea	108	2005	Non-comm.
Australian Future Fund	Australia	99	2006	Non-comm.
National Development Fund of Iran	Iran	91	2011	Oil & gas
National Welfare Fund	Russia	72	2008	Oil
Libyan Investment Authority	Libya	66	2006	Oil
Kazakhstan National Fund	Kazakhstan	65	2000	Oil
Samruk-Kazyna JSC	Kazakhstan	61	2008	Non-comm.
Alaska Permanent Fund	US – Alaska	55	1976	Oil
Brunei Investment Agency	Brunei	40	1983	Oil
Texas Permanent School Fund	US – Texas	38	1854	Oil & other
Khazanah Nasional	Malaysia	35	1993	Non-comm.
Emirates Investment Authority	UAE – Federal	34	2007	Oil
State Oil Fund	Azerbaijan	33	1999	Oil
New Zealand Superannuation Fund	New Zealand	23	2003	Non-comm.
New Mexico State Investment Council	US – New Mexico	20	1958	Oil & gas
State General Reserve Fund	Oman	18	1980	Oil & gas
Permanent University Fund	US – Texas	17	1876	Oil & gas
Timor-Leste Petroleum Fund	East Timor	17	2005	Oil & gas
Reserve Fund	Russia	16	2008	Oil
Social and Economic Stabilization Fund	Chile	15	2007	Copper

Table 1: World's largest sovereign wealth funds by assets, country, year of inception and origin of savings. Source: Sovereign Wealth Fund Institute database (July 2017 update). Estimates are at different time points, and the most recent are as of end of March, 2017. Notes: 1/ SAMA is the Saudi Arabian Monetary Authority. 2/ Based on the Sovereign Wealth Fund Institute estimates.

of assets to oil rents across the economies listed in Table 2 is quite telling of the variation in the importance of annual oil income relative to accumulated assets. For instance, in the case of Norway, annual rents from oil are rather insignificant when compared to the size of their SWF, which as we discuss later, has an important implication for their optimal portfolio choice. We shall see that given that Norway has a low ratio of oil income to accumulated assets, that means the component of hedging (against oil income shocks) in their portfolio allocation should be lower than, say, Saudi Arabia where annual oil income is still significant at it constitutes around a fifth of the state's accumulated wealth.

Another relevant dimension is the ratio of assets to proved oil reserves, which is reported in the last column of Table 2.[7] This ratio is a measure of above- to under-ground wealth, and it is

---

composed additionally of significant natural gas reserves.

[7]The value of proved oil reserves is calculated based on the stock of proved reserves at end of 2016 valued using

Country	SWF Assets	$\frac{\text{Assets}}{\text{GDP}}$	$\frac{\text{Oil rents}}{\text{GDP}}$	$\frac{\text{Assets}}{\text{Oil rents}}$	$\frac{\text{Assets}}{\text{Oil reserves}}$
UAE	1098	3.15	0.11	28.10	0.25
Norway	922	2.49	0.03	81.79	4.09
Saudi Arabia	697	1.08	0.23	4.82	0.06
Kuwait	524	4.59	0.38	11.93	0.12
Qatar	320	2.10	0.06	35.86	0.29
Russia	88	0.07	0.06	1.26	0.03
Kazakhstan	67	0.38	0.06	6.83	0.05

Table 2: World's largest sovereign wealth funds: Total assets in relation to economic attributes. Source: Sovereign Wealth Fund Institute database (July 2017 update). Notes: For SWF assets, all of the oil-based funds for each country are grouped together. The GDP data is obtained from the International Monetary Fund International Financial Statistics database and refer to 2016 figures, except for Kazakhstan and Kuwait where the GDP figure is for 2015. Oil rents are from the World Bank World Development Indicators database, and all figures are for 2015. Proved oil reserves (as of end of 2016) are from the U.S. Energy Information Administration.

evident that this ratio also exhibits large variation across these countries. In Norway, the bulk of its wealth is above-ground wealth as the state used oil revenue to build its SWF, the size of which is roughly 4 times the size of its proved reserves. For Qatar and the UAE, the accumulated assets above ground represent nearly a quarter of the remaining wealth underground. For Saudi Arabia, its pockets of underground wealth are much deeper in comparison as the country still holds about 16 percent of the world's crude oil reserves.

With regard to the funds' allocation across asset classes, Figure 1 shows the asset allocation for the five biggest oil-based SWFs. The figures indicate that the allocation strategies of these funds seem to have converged on an equity share that is around 60 percent of the total allocation. In the cases of Norway and Saudi Arabia, the remainder is mostly allocated to bonds with a negligible share going to other investments such as real estate and hedge funds. Kuwait and Qatar have both allocated a share to other investments (e.g. real estate, private equity in infrastructure and hedge funds) that is higher than that allocated to bonds. Their shares in the former class amounted to 26 percent for each country, while their allocation to bonds was 16 percent and 17 percent, respectively. For the UAE (ADIA), the allocation to bonds and other investments is roughly the same. This is likely to be a manifestation of more risk tolerance on the part of Kuwait, Qatar and the UAE (ADIA), as these funds seek the higher returns offered by nontraditional asset classes.

---

the average of Brent and WTI prices in 2016, which is US$ 44.16 per barrel.
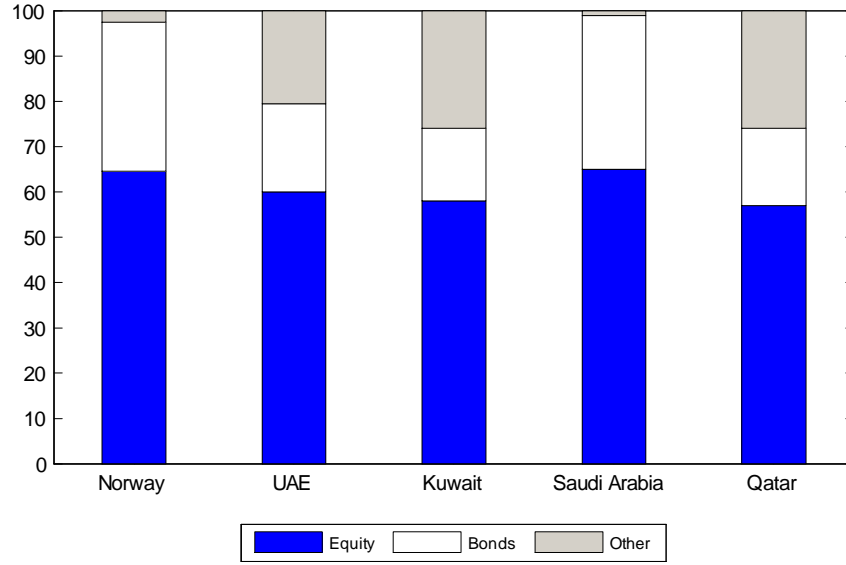
Figure 1: Asset allocation for large SWFs across asset classes. Source: Sovereign Wealth Fund Institute database (July 2017 update). Notes: The asset class "other" includes real estate, alternative investments, hedge funds, and private equity (particularly in infrastructure projects).

3   Modelling Framework

3.1   The Model

We consider a discrete-time model where F_t is the value of the fund at time t, i.e. at the beginning of the period $[t, t+1[$ and Y_t is the income from oil allocated to the fund at time t. Let C_t denote consumption out of the fund over the interval $[t, t+1[$ decided upon and effected at time t. The financial market consists of a risk-free asset and n risky assets. The continuously compounded returns on the risk-free and risky asset i, $i = 1, ..., n$, over the interval $[t, t+1[$ are denoted $R_{0,t}$ and $R_{i,t}$, respectively.

The fund's evolution over the period $[t, t+1[$ is given by

$$F_{t+1} = (F_t + Y_t - C_t) R_{F,t+1}, \tag{1}$$

where $R_{F,t+1}$ is the return on the fund's portfolio, which is determined by the return on the portfolio constituents as follows:

$$R_{F,t+1} = \pi_{0,t} R_{0,t+1} + \pi'_t R_{t+1} = R_{0,t+1} + \sum_{i=1}^{n} \pi_{i,t} \left(R_{i,t+1} - R_{0,t+1} \right), \tag{2}$$

where $\pi_{0,t}$ is the weight on the risk-free asset, $\pi_t = (\pi_{1,t}, ..., \pi_{n,t})'$, where $\pi_{i,t}$ denotes the weight on the i-th risky asset, with $R_{i,t+1} - R_{0,t+1}$ representing the excess returns on asset i. The

fund manager's objective is to maximize the expected value of the stream of current and future (discounted) utility of consumption, subject to a discount rate $0 < \delta < 1$:

$$\max_{\{C_t, \pi_t\}_{t=0}^{\infty}} \mathrm{E}\left[\sum_{t=0}^{\infty} \delta^t U\left(C_t\right)\right],$$

where $\mathrm{E}\left[\cdot\right]$ denotes the expectation operator. We assume the fund manager's utility is given by the constant relative risk aversion (CRRA) utility:

$$U\left(C_t\right) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{3}$$

where $\gamma > 0$, $\gamma \neq 1$, is the coefficient of relative risk aversion, and $\frac{1}{\gamma}$ is the elasticity of intertemporal substitution (EIS).[8]

Define the vector of log excess returns

$$x_{t+1} = \begin{pmatrix} r_{1,t+1} - r_{0,t+1} \\ r_{2,t+1} - r_{0,t+1} \\ \vdots \\ r_{n,t+1} - r_{0,t+1} \end{pmatrix},$$

where $r_{j,t+1} = \ln\left(R_{j,t+1}\right)$ for $j = 0, 1, ..., n$. We allow the model to include other state variables s_{t+1}, and we define the state vector as

$$z_{t+1} = \begin{pmatrix} r_{0,t+1} \\ x_{t+1} \\ s_{t+1} \end{pmatrix}.$$

We assume a VAR(1) model for z_{t+1}:

$$z_{t+1} = \Phi_0 + \Phi_1 z_t + \nu_{t+1}, \tag{4}$$

where the random vector ν_{t+1} represents shocks to the state variables. We assume that

$$\nu_{t+1} | \mathcal{F}_t \sim MVN\left(0, \Sigma_\nu\right),$$

with

$$\Sigma_\nu = Var_t(\nu_{t+1}) = \begin{pmatrix} \sigma_0^2 & \sigma_{0x}' & \sigma_{0s}' \\ \sigma_{0x} & \Sigma_{xx} & \Sigma_{xs}' \\ \sigma_{0s} & \Sigma_{xs} & \Sigma_{ss} \end{pmatrix},$$

where $\mathcal{F}_t$ is the time t information set.[9] The oil income that is added to the fund, Y_t, grows according to the following dynamics:

$$Y_{t+1} = Y_t \exp\left\{g + \xi_{t+1}\right\}, \tag{5}$$

---

[8] In the limiting case when $\gamma = 1$, it specializes to the log utility function.

[9] We maintain the assumption of homoskedasticity of the innovation ν_{t+1}, i.e. constant variances and covariances. As shown by Harvey (1989,1991) and Glosten et al. (1993) in the context of optimal asset allocation, predicting the second moment of returns provides smaller gains compared to predicting the first moment.

where g is the constant rate of growth, and the oil income shock is ξ_{t+1}. The oil income shock (ξ_{t+1}) and the shocks to the state variables (ν_{t+1}) are assumed to be correlated where we have

$$\xi_{t+1} = \beta' \nu_{t+1} + \sigma_{(o)}\xi_{t+1}^{(o)}, \tag{6}$$

with $\xi_{t+1}^{(o)}$ and ν_{t+1} assumed independent, and $\xi_{t+1}^{(o)} \sim N(0,1)$. With the decomposition in (6), shocks to oil income can be induced by shocks to the state variables via the vector β, or due to an idiosyncratic oil shock $\xi_{t+1}^{(o)}$ with variance $\sigma_{(o)}^2$.

3.2 Optimal Solution

For this problem, The Euler equation for consumption is given by

$$\mathrm{E}_t \left[\left\{ \delta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right\} R_{i,t+1} \right] = 1 \tag{7}$$

for $i = 1, 2, ..., n, F$. To solve for the optimal weights $\pi_{i,t}$, we adopt the log-linear approximation method of Campbell (1993). One can write the budget constraint in (1) as

$$\frac{F_{t+1}}{Y_{t+1}} = \left(\frac{F_t}{Y_t} + 1 - \frac{C_t}{Y_t} \right) \frac{Y_t}{Y_{t+1}} R_{F,t}. \tag{8}$$

This is equivalent in logs to

$$f_{t+1} - y_{t+1} = log\left(1 + exp\{f_t - y_t\} - exp\{c_t - y_t\}\right) - \Delta y_{t+1} + r_{F,t+1}. \tag{9}$$

The first term on the right-hand side is a nonlinear function of the wealth-to-income and consumption-to-income ratios. Consider this expression as a function of two variables $z_t^{(k)} = \ln(Z_t^{(k)})$, $k = 1, 2$, and take a first-order Taylor expansion around the means $\bar{z}^{(k)}$. The resulting approximation is

$$\ln\left(1 + exp(z_t^{(1)}) - exp(z_t^{(2)})\right) \approx \ln\left(1 + exp(\bar{z}^{(1)}) - exp(\bar{z}^{(2)})\right) + \frac{exp(\bar{z}^{(1)})}{1 + exp(\bar{z}^{(1)}) - exp(\bar{z}^{(2)})}\left(z_t^{(1)} - \bar{z}^{(1)}\right)$$
$$- \frac{exp(\bar{z}^{(2)})}{1 + exp(\bar{z}^{(1)}) - exp(\bar{z}^{(2)})}\left(z_t^{(2)} - \bar{z}^{(2)}\right).$$

We use this expansion for $z_t^{(1)} = f_t - y_t$ and $z_t^{(2)} = c_t - y_t$, and so we obtain the following log-linear approximation to the budget constraint:

$$f_{t+1} - y_{t+1} \approx k + \rho_f(f_t - y_t) - \rho_c(c_t - y_t) - \Delta y_{t+1} + r_{F,t+1}, \tag{10}$$

where

$$\rho_f = \frac{exp\{\mathrm{E}[f_t - y_t]\}}{1 + exp\{\mathrm{E}[f_t - y_t]\} - exp\{\mathrm{E}[c_t - y_t]\}}, \tag{11}$$

$$\rho_c = \frac{exp\{E[c_t - y_t]\}}{1 + exp\{E[f_t - y_t]\} - exp\{E[c_t - y_t]\}}, \tag{12}$$

and

$$k = -(1 - \rho_f + \rho_c) \ln(1 - \rho_f + \rho_c) - \rho_f \ln(\rho_f) + \rho_c \ln(\rho_c).$$

Campbell and Viceira (1999) derive an approximation for the log return on the fund as follows

$$r_{F,t+1} = r_{0,t+1} + \pi_t' x_{t+1} + \frac{1}{2} \pi_t' (\sigma_x^2 - \Sigma_{xx} \pi_t), \tag{13}$$

where $\sigma_x^2 = diag(\Sigma_{xx})$. The Euler equation in (7) is also nonlinear and a log-linear approximation is derived as follows (see Campbell et al. (2003) for details):

$$0 = \ln(\delta) - \gamma E_t[c_{t+1} - c_t] + E_t[r_{i,t+1}] + \frac{1}{2} Var_t[r_{i,t+1} - \gamma(c_{t+1} - c_t)]. \tag{14}$$

Proposition 1 *Under the log-linear approximation, the optimal portfolio allocation and log consumption are given by*

$$\pi_t = A_0 + A_1 z_t, \tag{15}$$

and

$$c_t = a + bf_t + (1 - b) y_t + B_1' z_t + z_t' B_2 z_t, \tag{16}$$

where A_0, A_1, a, b, B_1 and B_2 are given in Appendix A.

The proof of Proposition 1 is given in Appendix A. To understand this solution, it is instructive to study the solution in (15) and (16) further. We start with the optimal asset allocation in Section 3.3, followed by the optimal path for consumption in Section 3.4.

3.3   Optimal Asset Allocation

The parameters of the optimal asset allocation A_0 and A_1 can be decomposed as follows (see Appendix A for details):

$$\underbrace{A_0}_{Total} = \underbrace{A_0^{(s)}}_{Speculative} + \underbrace{A_0^{(h)}}_{Normal\ hedging} + \underbrace{A_0^{(o)}}_{Oil\ hedging},$$

$$\underbrace{A_1}_{Total} = \underbrace{A_1^{(s)}}_{Speculative} + \underbrace{A_1^{(h)}}_{Normal\ hedging}.$$

We first explain the distinction between the different components of the asset allocation based on how it serves the investor in achieving an optimal allocation and path for consumption. The demand for risky assets in this context aims to serve three motives: *speculative* demand, *normal*

hedging demand given financial market risk, and *oil hedging* demand. The expressions for the different demand components in $A_0 = A_0^{(s)} + A_0^{(h)} + A_0^{(o)}$ are given by

$$A_0^{(s)} = \frac{1}{b\gamma} \Sigma_{xx}^{-1} \left[H_x \Phi_0 + \frac{1}{2}\sigma_x^2 + \sigma_{0x} \right] - \sigma_{0x},$$

$$A_0^{(h)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[-\Lambda_0^{(z)} \right],$$

$$A_0^{(o)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[-(1-b)H_x \beta' \Sigma_\nu - \Lambda_0^{(y)} \right].$$

The term $A_0^{(s)}$ represents speculative demand due to the appearance of the coefficient of relative risk aversion γ as a scaling factor, where a larger γ generally reduces the demand for risky assets. Also, note that $A_0^{(s)}$ is independent of the parameters of the z_t process (Φ_0 and Φ_1), the parameters of the oil income process (g and $\sigma_{(o)}^2$), and also the correlations between the oil income and financial shocks, which is governed by β. The normal hedging demand is given by $A_0^{(h)}$ since it depends on the dynamics of z_t through $\Lambda_0^{(z)}$, while $A_0^{(o)}$ represents the component of demand that relates to hedging against oil income volatility via $\Lambda_0^{(y)}$, and also hedging against the correlation between oil income shocks and innovations to the financial market variables through β. Note that the oil hedging component $A_0^{(o)}$ will be zero when $\beta = 0$, which implies no correlation between oil income and the returns on the risky assets, since in such a case we also have $\Lambda_0^{(y)} = 0$.

For $A_1 = A_1^{(s)} + A_1^{(h)}$, we have the following decomposition into *speculative* and *normal hedging* demands:

$$A_1^{(s)} = \frac{1}{b\gamma} \Sigma_{xx}^{-1} \left[H_x \Phi_1 \right],$$

$$A_1^{(h)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[-\Lambda_1 \right].$$

A similar argument applies to $A_1^{(s)}$, which captures speculative demand due to the presence of γ. Normal hedging demand is given by $A_1^{(h)}$ due to the presence of Λ_1, which is a function of the parameters of the z_t process, and is independent of oil income shocks and their correlation with the financial market variables. By inspecting the term Λ_1 in the appendix, we can see that when $\Phi_1 = 0$, we have $\Lambda_1 = 0$ and consequently $A_1^{(h)}$ becomes zero. This is intuitive since $\Phi_1 = 0$ implies no variation in the available investment opportunities, so the financial hedging component should be zero.

3.4 Optimal Path for Consumption

With regard to the optimal path for consumption given by (16), we have an autonomous level of consumption a that is independent of the path of f_t, y_t or z_t. This level can be decomposed

as follows:

$$a = a^{(y)} + a^{(z)},$$

with

$$a^{(y)} = \frac{1}{\rho_f - 1} \left[(1-b)g - V_0^{(y)} + B_1^{(y)'} \Phi_0 \right],$$

$$
\begin{aligned}
a^{(z)} &= \frac{1}{\rho_f - 1} \left[bk - \frac{1}{\gamma} \ln(\delta) + \left(1 - \frac{1}{\gamma}\right) \Gamma_0 - V_0^{(z)} \right] \\
&\quad + \frac{1}{\rho_f - 1} \left[B_1^{(z)'} \Phi_0 + \Phi_0' B_2 \Phi_0 + vec(B_2)' vec(\Sigma_\nu) \right],
\end{aligned}
\tag{17}
$$

where the superscripts in $a^{(y)}$ and $a^{(z)}$ signify that they depend chiefly on the parameters of the oil income process and the state variables' dynamics, respectively. We have $a^{(y)}$ representing the autonomous level of consumption that increases with a higher rate of growth in oil income (g), and decreases with $V_0^{(y)}$ which depends on the variance in oil income ($\sigma_{(o)}^2$), among other parameters. It also depends on how the shocks to oil income correlate with shocks to the state variables through the term $B_1^{(y)'} \Phi_0$ which is non-zero when $\beta \neq 0$. It is worth noting that the sensitivity of $a^{(y)}$ with respect to g depends on the value of b, and the latter plays a significant role in the path of optimal consumption as we discuss below in detail. The other component of the autonomous level of consumption is $a^{(z)}$ which depends primarily on the parameters of the z_t process.

We now turn to the role that b plays in influencing the path of optimal consumption. First note that b represents the marginal propensity to consume out of accumulated wealth, thus it is non-negative at the optimal path.[10] Thus, ignoring a and the terms in z_t, we can see from (16) that optimal consumption is a weighted average of f_t and y_t, with respective weights b and $1 - b$. This finding is consistent with the results in Veceira (2001) which are derived from a model based on CRRA utility, and which deals explicitly with retirement horizon effects for investors.

Recall that $b = \frac{\rho_f - 1}{\rho_c}$, with ρ_f and ρ_c respectively given by (11) and (12). Thus b is a function of two optimal ratios: the wealth-to-income ratio $f_t - y_t$, and the consumption-to-income ratio $c_t - y_t$. Both ratios appear in expectation, thus one can think of b as capturing the steady-state values of these two ratios. Intuitively, for a country that is dependent on a volatile source of income, it is optimal to build a large wealth buffer such that the path of consumption is relatively insulated from income volatility. As the ratio of wealth-to-income ($f_t - y_t$) becomes larger and larger, we have b approaching 1. This case is discussed in detail in Section 3.5.1. This case is

---

[10]It is straightforward to show that a negative b implies that the mean consumption-to-income ratio exceeds the mean wealth-to-income ratio, which would lead to an eventual exhaustion of the fund.

likely to occur in a country where the SWF existed for a long period of time, and in which no significant drawdowns have been made.[11]

In continuing the analogy with Veceira (2001), we note that in his model $b = 1$ for retired investors, i.e. those investors whose labor income is zero. In the context of SWFs, the similarity is not with respect to the state of retirement since SWFs are by construction meant to last for generations to come. The analogy however is relevant when thinking about the progress made so far by oil-dependent economies with regard to saving sufficient wealth relative to their annual stream of oil income. As shown in Table 2, Norway and the UAE are two countries where the ratio of wealth-to-income has become substantially higher than other major oil exporters owning SWFs, and thus they are closer to achieving the model-implied steady-state value for b which is 1. For a country such as Saudi Arabia or Russia, b would be smaller in the short term, and that means that the weight on y_t in (16) will be large implying large fluctuations in consumption as oil income fluctuates. The recent experience of Saudi Arabia and Russia during the oil revenue slump in 2015-2017 corroborates our analysis. The remaining terms in (16) involve z_t, thus they represent the change in optimal consumption given variation in the investment opportunities.

3.5 Special Cases of Interest

3.5.1 Mature Funds

Continuing our discussion in Section 3.4 with regard to how b controls the optimal path for consumption, we now analyze the solution in the limiting case of $b = 1$. It is straightforward to show that letting $b \to 1$ is equivalent to

$$\frac{1}{1 + exp\{\mathrm{E}[f_t - y_t]\} - exp\{\mathrm{E}[c_t - y_t]\}} \to 0.$$

This happens when the ratio of wealth to income $(f_t - y_t)$ increases as would be the case with a mature fund in the sense that the fund value has become so large such that the annual stream of oil income is negligible in comparison. In this limiting case, we have $\Omega_1^{(y)} = V_1^{(y)} = B_1^{(y)} = 0$ implying

$$V_0^{(y)} = \frac{\gamma}{2}\sigma_{(o)}^2.$$

In this case, the optimal consumption is given by

$$c_t - f_t = a^{(y)} + a^{(z)} + B_1^{(z)'}z_t + z_t'B_2z_t,$$

---

[11]Indeed our empirical results reported in Section 6 indicate that for the large majority of different combinations of parameter values, we have $b \to 1$. The exceptions to this result occur only for parameter configurations which deviate substantially from what is implied by the historical data, or for nonstandard levels of risk aversion and oil income volatility.

where

$$a^{(y)} = \frac{1}{\rho_f - 1} \left[-\frac{\gamma}{2} \sigma^2_{(o)} \right],$$

and $a^{(z)}$ is given by (17) with b set to 1.

When the value of y_t is much smaller than the value of accumulated wealth, the ratio of consumption to wealth $(c_t - f_t)$ becomes constant if there is no variation in investment opportunities, that is no variation in z_t. In such a case, consumption out of wealth has a unit elasticity. The autonomous level of consumption, $a = a^{(y)} + a^{(z)}$, under the optimal path for consumption still depends on income through $a^{(y)}$, which in turn depends on y_t through $\sigma^2_{(o)}$. So the only impact of y_t on optimal consumption is driven by the variance of oil income, which has a negative impact on c_t and the magnitude of its effect scales with γ. Thus, the investor reacts to the high volatility of income by reducing her current consumption out of wealth. Finally as $b \to 1$, we have $A_0^{(o)} \to 0$, which means the oil hedging demand component in the optimal π_t will be zero.

3.5.2 Zero Correlation between Oil Income and the State Variables

When $\beta = 0$ in (6), there is no correlation between oil income shocks and the innovations to the state variables. In such a case, we also have $\Omega_1^{(y)} = V_1^{(y)} = B_1^{(y)} = 0$, implying

$$V_0^{(y)} = \frac{\gamma}{2} \sigma^2_{(o)}.$$

Similar to the previous case of a mature fund, the optimal allocation π_t changes as the oil hedging component $A_0^{(o)}$ becomes zero. For c_t, it still depends on the oil income and the variance of its shocks as follows:

$$c_t = a^{(y)} + a^{(z)} + b f_t + (1 - b) y_t + B_1^{(z)'} z_t + z_t' B_2 z_t,$$

where

$$a^{(y)} = \frac{1}{b \rho_c} \left[(1 - b) g - \frac{\gamma}{2} \sigma^2_{(o)} \right].$$

If the oil income and the state variables are uncorrelated, the optimal consumption decreases in the volatility of income, and increases in expected income growth. The impact of g is important when b is small, that is when the ratio of wealth to income is small. As the fund matures, implying $b \to 1$, the impact of the growth rate of income vanishes, but the negative impact of volatility remains.

4 Parameter Estimation

We assume $\nu_{t+1}|\mathcal{F}_t \sim MVN(0, \Sigma_\nu)$ and $\xi_{t+1}^{(o)} \sim N(0,1)$. Let d denote the dimension of the innovation vector ν_{t+1}. Also let θ_1 denote the true parameter vector for the VAR process in (4), such that $\theta_1 = \left(\Phi_0', vec(\Phi_1)', vech(\Sigma_\nu)\right)'$, where the vec operator stacks the columns of Φ_1 into a $(d^2 \times 1)$ vector, and the $vech$ operator stacks the lower triangular part including the main diagonal of Σ_ν into a $\left(\frac{d(d+1)}{2} \times 1\right)$ vector. Similarly, let $\theta_2 = \left(g, \beta', \sigma_{(o)}^2\right)$ denote the true parameter vector for (6), and let $\theta = \left(\theta_\nu', \theta_\xi'\right)'$ be the true full parameter vector.

This implies the following t-th period log-likelihoods:

$$
l_{\nu,t}(\theta_\nu) = -\frac{1}{2}\left[d\ln(2\pi) + \ln|\Sigma_\nu| + (z_t - \Phi_0 - \Phi_1 z_{t-1})' \Sigma_\nu^{-1} (z_t - \Phi_0 - \Phi_1 z_{t-1})\right],
$$

$$
l_{\xi,t}(\theta_\nu, \theta_\xi) = -\frac{1}{2}\left[\ln(2\pi) + \ln\sigma_{(o)}^2 + \frac{\xi_t^2}{\sigma_{(o)}^2}\right],
$$

where

$$
\xi_t = y_t - y_{t-1} - g - \beta'\nu_t = y_t - y_{t-1} - g - \beta'(z_t - \Phi_0 - \Phi_1 z_{t-1}).
$$

Note the dependence of $l_\xi(\cdot)$ on θ_ν since ξ_t depends on ν_t. The quasi-maximum likelihood estimator is $\widehat{\theta} = \left(\widehat{\theta}_\nu', \widehat{\theta}_\xi'\right)'$ where

$$
\widehat{\theta} = \underset{\theta \in \Theta}{\arg\max}\, T^{-1}\sum_{t=1}^{T} l_t(\theta) \tag{18}
$$

$$
= \underset{\theta \in \Theta}{\arg\max}\, T^{-1}\sum_{t=1}^{T} l_{\nu,t}(\theta_\nu) + l_{\xi,t}(\theta_\nu, \theta_\xi). \tag{19}
$$

Define the score vectors $s_{\nu,t}(\theta_\nu)$ and $s_{\xi,t}(\theta_\nu, \theta_\xi)$, and the combined score vector $s_t(\theta) = (s_{\nu,t}(\theta_\nu), s_{\xi,t}(\theta_\nu, \theta_\xi))$. Under standard regularity conditions (e.g. Newey and McFadden (1994)), we have

$$
\sqrt{T}\left(\widehat{\theta} - \theta_0\right) \xrightarrow{d} N(0, \mathcal{I}^{-1}\mathcal{J}(\mathcal{I}^{-1})'),
$$

where

$$
\mathcal{J} = \text{Var}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} s_t(\theta)\right],
$$

$$
\mathcal{I} = -\text{E}\left[\frac{\partial s_t(\theta)}{\partial\theta}\right] = -\text{E}\begin{bmatrix} \frac{\partial l_{\nu,t}(\theta_\nu)}{\partial\theta_\nu\partial\theta_\nu'} & 0 \\ \frac{\partial l_{\xi,t}(\theta_\xi)}{\partial\theta_\xi\partial\theta_\nu'} & \frac{\partial l_{\xi,t}(\theta_\xi)}{\partial\theta_\xi\partial\theta_\xi'} \end{bmatrix}.
$$

In principle, (18) can be estimated in one step, however given the potentially large number of parameters,[12] estimation efficiency may be adversely affected if the sample size is small.

---

[12] Note that θ_ν depends on the dimension of the VAR model in (4). In general, the number of parameters in θ_ν is $\frac{3}{2}d(d+1)$, while the dimension of θ_ξ is $d+2$.

In addition, the log-likelihood function may be flat near the optimum which again results in lower estimation efficiency. For this reason, we use the maximization by parts (MbP) algorithm proposed in Song et al. (2005). The structure of the log-likelihood function for our model is convenient for the use of this algorithm.

The MbP algorithm iterates over different components of the log-likelihood until convergence is achieved. Given the log-likelihood

$$l_t(\theta) = l_{\nu,t}(\theta_\nu) + l_{\xi,t}(\theta_\nu, \theta_\xi),$$

the corresponding score equations are

$$\begin{pmatrix} \frac{\partial l_{\nu,t}(\theta_\nu)}{\partial \theta'_\nu} + \frac{\partial l_{\xi,t}(\theta_\nu, \theta_\xi)}{\partial \theta'_\nu} \\ \frac{\partial l_{\xi,t}(\theta_\nu, \theta_\xi)}{\partial \theta'_\xi} \end{pmatrix} = 0.$$

Note that solving $\frac{\partial l_{\xi,t}(\theta_\nu, \theta_\xi)}{\partial \theta'_\xi} = 0$ is easy since θ_ξ is of low dimension. Let $\theta_i^{(k)}$ denote the k-th iteration for the estimator, $i = (\nu, \xi)$. The algorithm is specified in what follows:

- Step 1. Solve $\frac{\partial l_{\nu,t}(\theta_\nu)}{\partial \theta'_\nu} = 0$ for $\theta_\nu^{(1)}$, and $\frac{\partial l_{\xi,t}(\theta_\nu^{(1)}, \theta_\xi)}{\partial \theta'_\xi} = 0$ for $\theta_\xi^{(1)}$.

- Step 2. For $k = 2, 3, ...$, solve $\frac{\partial l_{\nu,t}(\theta_\nu)}{\partial \theta'_\nu} = -\frac{\partial l_{\xi,t}(\theta_\nu^{(k-1)}, \theta_\xi^{(k-1)})}{\partial \theta'_\nu}$ for $\theta_\nu^{(k)}$, and $\frac{\partial l_{\xi,t}(\theta_\nu^{(k-1)}, \theta_\xi)}{\partial \theta'_\xi} = 0$ for $\theta_\xi^{(k)}$.

Song et al. (2005) show that $\theta^{(k)} = \left(\theta_\nu^{(k)}, \theta_\xi^{(k)}\right)$ is consistent for each k, and that $\theta^{(k)}$ converges to the QMLE $\widehat{\theta}$ under standard regularity condition. The asymptotic variance of $\theta^{(k)}$ is computed using the formulas provided in Theorem 3 in Song et al. (2005).

5 Empirical Analysis

5.1 Data

To calibrate the model, we use annual data over the period 1976 to 2016. For oil income, we use the net government cash flow from petroleum for Norway. We use the risk-free asset is proxied by the U.S. 3-month Treasury bill rate. We consider two asset classes: equity and bonds. For equity, we use the returns on the S&P500 index as a proxy for global equity returns, and for bonds, we use the 10-year Treasury constant maturity rate. For the state variables, we use the dividend yield for the S&P500 as a predictor of future equity returns; see Campbell and Shiller (1988) and Fama and French (1988).[13]

---

[13]Other state variables that are commonly used in the literature are the term spread (i.e. the yield spread between long-term and short-term bonds), and the credit spread (proxied by the difference between Baa corporate bond yield and the 10-year Treasury constant maturity rate). The model allows for including additional variables, however we include only the dividend yield to maintain parsimony in the VAR models presented below.

Variable	Mean	St. Dev.	Skew.	Kurt.	Min	Max	$\rho(1)$
Return on risk-free asset $(r_{rf,t})$	0.047	0.035	0.533	2.898	0.000	0.143	0.890
Excess returns on equity $(\widetilde{r}_{eq,t})$	0.078	0.159	-0.742	3.288	-0.381	0.321	-0.038
Excess returns on LT bonds $(\widetilde{r}_{bn,t})$	0.031	0.099	0.033	2.035	-0.142	0.218	-0.208
Dividend yield (d_t)	0.029	0.013	0.624	2.150	0.011	0.056	0.918
Net oil cashflow, US\$ bn. (Y_t)	26.074	25.001	0.997	2.677	0.809	88.902	0.894

Table 3: Summary statistics for the variables included in the empirical analysis for the sample period 1976-2016. $\rho(1)$ is the autocorrelation coefficient at lag 1.

The data for net government cash flow from petroleum is obtained from the Norwegian Ministry of Finance and Statistics Norway. The data for U.S. 3-month Treasury bill rate and 10-year Treasury constant maturity rate are obtained from the FRED database at the Federal Reserve Bank of St. Louis. The S&P 500 index and the dividend yield data are obtained from Center for Research in Security Prices (CRSP) data files.

Table 3 provides some descriptive statistics. The risk-free asset had a mean return of 4.7 percent over the sample period with a standard deviation of 3.5 percent. The mean excess return on equity and bonds recorded 7.8 and 3.1 percent, respectively, with equity subject to higher volatility. Equity also exhibits negative skewness and higher kurtosis relative to bonds. The dividend yield was 2.9 percent on average during the sample period with a standard deviation of 1.3 percent. The return on the risk-free asset is more persistent, relative to the excess returns on equity and bonds, with the former having an autocorrelation coefficient $\rho(1)$ equal to 0.89.

The Norwegian state's net oil cash flow was US\$ 26.1 billion on average, however it has quite a sizable standard deviation as it started from a meagre amount (close to US\$ 0.8 billion) in 1976 and increased progressively with the increase in both oil prices and production, recording a peak in 2008 as it reached US\$ 88.9 billion. It is worth noting that Norway created its SWF in 1990, which was called the Government Petroleum Fund, before its name was changed to Government Pension Fund (Global) in 2006. The first transfer to the fund was made in 1996 and the amount was around US\$ 0.4 billion. Although the Norwegian fund started later than the start of the sample period, we still use the data on the government net cash flow from oil as a reliable time series to estimate the average rate of growth and the volatility of the stream of oil income that is deposited into the fund.

5.2 Estimation Results

In this model, we have z_t in (4) defined as

$$z_t = \begin{pmatrix} r_{rf,t} \\ \widetilde{r}_{eq,t} \\ \widetilde{r}_{bn,t} \\ d_t \end{pmatrix} = \begin{pmatrix} r_{rf,t} \\ r_{eq,t} - r_{rf,t} \\ r_{bn,t} - r_{rf,t} \\ d_t \end{pmatrix}.$$

In the VAR(1) model $z_{t+1} = \Phi_0 + \Phi_1 z_t + \nu_{t+1}$, the estimates of Φ_0 and Φ_1 are given by (standard errors reported in brackets):

$$\widehat{\Phi}_0 = \begin{bmatrix} -0.0046 \\ {\scriptstyle(0.0078)} \\ 0.0166 \\ {\scriptstyle(0.1819)} \\ 0.0846 \\ {\scriptstyle(0.0695)} \\ 0.0011 \\ {\scriptstyle(0.0051)} \end{bmatrix}, \quad \widehat{\Phi}_1 = \begin{bmatrix} 0.7564 & 0.0245 & -0.0725 & 0.5295 \\ {\scriptstyle(0.1576)} & {\scriptstyle(0.0205)} & {\scriptstyle(0.0336)} & {\scriptstyle(0.3068)} \\ -1.9249 & 0.0103 & 0.0697 & 5.1262 \\ {\scriptstyle(2.5754)} & {\scriptstyle(0.3733)} & {\scriptstyle(0.4453)} & {\scriptstyle(8.0933)} \\ 0.9120 & -0.0525 & -0.2314 & -2.9281 \\ {\scriptstyle(1.2731)} & {\scriptstyle(0.1771)} & {\scriptstyle(0.2995)} & {\scriptstyle(3.1414)} \\ 0.0020 & 0.0064 & -0.0033 & 0.9265 \\ {\scriptstyle(0.0873)} & {\scriptstyle(0.0115)} & {\scriptstyle(0.0162)} & {\scriptstyle(0.2610)} \end{bmatrix}.$$

For $r_{rf,t+1}$, its own lag is a significant predictor with a positive coefficient due to the evident persistence in the risk-free rate of return. The current value of $\widetilde{r}_{bn,t}$ is also a significant predictor with a negative coefficient implying that rising returns on longer term bonds lead to a decline in returns on short-term bond returns due to the impact of investors shifting their maturity preferences. For $\widetilde{r}_{eq,t+1}$ and $\widetilde{r}_{bn,t+1}$, none of the predictors are significant. For d_{t+1}, its own lagged value is the only significant predictor due to its strong persistence. These results are largely consistent with the literature; see, for example, Campbell et al. (2003). The notable exception is that the dividend yield is often found to be a significant predictor of future equity returns. In our results, the coefficient has the correct sign however its large standard error is likely due to the relatively short sample period. With regard to the stability of the VAR(1) process in (4), all of the eigenvalues of Φ_1 are inside the unit circle, with the largest eigenvalue equal to 0.93 indicating a stationary process that mean reverts to the unconditional mean vector μ_z an estimate of which is given by $\left(I - \widehat{\Phi}_1 \right)^{-1} \widehat{\Phi}_0 = \begin{pmatrix} 0.024 & 0.080 & 0.034 & 0.021 \end{pmatrix}'$.

The estimate of the residual variance-covariance matrix (Σ_ν) is

$$\widehat{\Sigma}_\nu = \begin{bmatrix} 0.0001 \\ {\scriptstyle(0.0000)} \\ -0.0001 & 0.0223 \\ {\scriptstyle(0.0007)} & {\scriptstyle(0.0092)} \\ -0.0006 & 0.0004 & 0.0084 \\ {\scriptstyle(0.0003)} & {\scriptstyle(0.0054)} & {\scriptstyle(0.0030)} \\ 0.0000 & -0.0006 & -0.0001 & 0.0000 \\ {\scriptstyle(0.0000)} & {\scriptstyle(0.0003)} & {\scriptstyle(0.0002)} & {\scriptstyle(0.0000)} \end{bmatrix},$$

	Parameter estimate
β_1	20.4379
	(9.2431)
β_2	-0.5206
	(1.3834)
β_3	1.0889
	(1.1566)
β_4	-14.7282
	(49.4780)
g	0.0464
	(0.1048)
$\sigma^2_{(o)}$	0.2487
	(0.0630)

Table 4: Parameter estimates for equations (5) and (6), which describe the dynamics of oil income growth, and the interaction between oil income shocks and innovations to the financial market variables. Estimates are obtained using annual data for the sample period 1976-2016, with standard errors reported in parentheses.

which implies the following estimate of the matrix of cross correlations (R_ν):

$$\widehat{R}_\nu = \begin{bmatrix} 1.0000 & & & \\ -0.0459 & 1.0000 & & \\ -0.5795 & 0.0290 & 1.0000 & \\ 0.2069 & -0.8762 & -0.1252 & 1.0000 \end{bmatrix}.$$

The estimates of θ_ξ are given in Table (4), with standard errors reported in brackets. Despite the lack of statistical significance with the exception of β_1, the signs of the beta coefficients indicate how shocks to the state variables (namely the returns on the risk-free asset and the excess returns on equity and bonds) correlate with the shocks to income from oil. Recalling the relationship in (6) where we have $\xi_{t+1} = \beta' \nu_{t+1} + \sigma_{(o)} \xi_{t+1}^{(o)}$, with ξ_{t+1} denoting the shock to oil income, we can see shocks to equity returns (and the dividend yield) being negatively correlated with oil income shocks, while shocks to bond returns (whether for short-term bonds captured by the risk-free asset, or for longer-term bonds) tend to have a positive correlation with shocks to the income from oil. The estimate of oil income growth is 4.64 percent with a variance of 24.87. This rather large variance is consistent with the in-sample variance of the growth in oil income $\ln(Y_t) - \ln(Y_{t-1})$, which is 29.63 percent, which is driven by excessive price volatility during the sample period.

6 Optimal Solution: An Empirical Example

In this section, we calibrate the model using the parameter estimates presented above, and use it to study the impact of changes in the behavioral parameters γ and δ, in addition to changes in β and $\sigma^2_{(o)}$, on the optimal allocation and consumption path. It is clear from (15) that the optimal asset allocation is a linear function of the state vector z_t, therefore the optimal allocation changes over time in response to variation in the investment opportunities, however we can study the

mean allocation given by $\mathrm{E}[\pi_t] = A_0 + A_1 \mathrm{E}[z_t]$. Table 5 shows how the optimal asset allocation changes with different levels of risk aversion (γ), rate of time preference (δ) and the variance to oil income shocks $(\sigma^2_{(o)})$. As the level of risk aversion increases in the range of $\gamma \in [3, 12]$, the investor tends to decrease her holdings of risky assets (equity and bonds) where the total demand generally declines. In parallel, the investor also decreases her leverage as the short position in cash (i.e. the risk-free asset) is also declining. However, in contrast to the general decline in the total demand for equity, the sub-component of total demand that relates to intertemporal hedging increases. This occurs as a higher γ leads the investor to increase her reliance on equity as a good hedge especially given its negative correlation with oil income shocks. In fact, for very high levels of risk aversion, the demand for equity for normal hedging purposes becomes so large that it increases the total demand for equity; compare the third and fourth panels of Table 5. As for bonds, the total demand keeps decreasing as γ increases since bonds have a positive correlation with oil income shocks.

As the rate of time preference δ gets larger, implying a larger weight on the utility of future consumption, the total demand for equity increases while that for bonds declines. The change in total leverage is marginal, especially for low values of γ. This indicates that an investor with a relatively long horizon will opt for a higher share in equity and a larger equity-to-bond ratio as she capitalizes on the higher excess return on stocks.

To study the impact of oil income volatility, we assume a fixed $\delta = 0.95$, and vary the level of $\sigma^2_{(o)}$ by scaling it by the factors reported in the table header: 0.85, 1.00 and 1.15. Note that the middle column corresponds to the in-sample estimate of $\sigma^2_{(o)}$. As the variance of oil income shocks increase, the investor generally increases her demand for equity relative to bonds. This is driven by the negative correlation between shocks to oil income and innovations to the excess return on equity. Still, for most values of γ, the demand for equity induced by the oil hedging component is very low compared to total demand. However, for $\gamma = 12$, and given an amplified level of volatility for oil income shocks as reported in the last column, we see a significant increase in the demand for equity which is now driven by both components: the normal hedging demand and the oil hedging demand.[14]

---

[14] The reason the demand for hedging against oil appears negligible in most cases can be seen from its formula:

$$A_0^{(o)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[-(1-b) H_x \beta' \Sigma_\nu - \Lambda_0^{(y)} \right].$$

With b being 1 or very close to 1 at the optimal path for different values of γ and δ, the first term in square brackets is mostly zero. Also, the term $\Lambda_0^{(y)} = H_x \Sigma_\nu' B_1^{(y)}$ equals zero in the limit as $b \to 1$. This occurs since $B_1^{(y)} \to 0$ as $b \to 1$; see the discussion in Section 3.5.1.

	δ			$\sigma^2_{(o)}$		
	0.90	*0.95*	*0.99*	*0.85*	*1.00*	*1.15*
$\gamma = 3$						
Equity (total demand)	129.76	132.67	135.18	131.96	132.67	133.52
Equity (hedging, normal)	27.74	30.64	33.16	29.94	30.64	31.49
Equity (hedging, oil)	0.01	0.01	0.01	0.01	0.01	0.01
Bonds (total demand)	98.33	96.66	95.21	97.06	96.66	96.17
Bonds (hedging, normal)	-7.64	-9.31	-10.75	-8.90	-9.31	-9.79
Bonds (hedging, oil)	0.00	0.01	0.00	0.00	0.01	0.01
Cash	-128.09	-129.33	-130.39	-129.02	-129.33	-129.69
$\gamma = 6$						
Equity (total demand)	89.05	91.63	93.77	89.25	91.63	94.68
Equity (hedging, normal)	38.01	40.58	42.72	38.21	40.58	43.62
Equity (hedging, oil)	0.01	0.01	0.01	0.01	0.01	0.01
Bonds (total demand)	50.72	49.35	48.20	50.62	49.35	47.71
Bonds (hedging, normal)	-2.29	-3.67	-4.82	-2.40	-3.67	-5.31
Bonds (hedging, oil)	0.00	0.00	0.00	0.00	0.00	0.00
Cash	-39.77	-40.98	-41.97	-39.86	-40.98	-42.38
$\gamma = 9$						
Equity (total demand)	81.96	84.65	86.87	79.32	84.65	92.24
Equity (hedging, normal)	47.90	50.60	52.81	45.27	50.60	58.18
Equity (hedging, oil)	0.02	0.01	0.02	0.01	0.01	0.02
Bonds (total demand)	34.06	32.57	31.33	35.48	32.57	28.27
Bonds (hedging, normal)	-1.32	-2.80	-4.04	0.12	-2.80	-7.10
Bonds (hedging, oil)	0.00	0.00	0.00	0.00	0.00	0.00
Cash	-16.02	-17.22	-18.20	-14.48	-17.22	-21.18
$\gamma = 12$						
Equity (total demand)	87.93	91.67	94.92	79.82	91.67	123.36
Equity (hedging, normal)	62.36	66.09	69.29	54.26	66.09	85.28
Equity (hedging, oil)	0.02	0.04	0.07	0.02	0.04	9.53
Bonds (total demand)	21.00	18.77	16.81	25.70	18.77	-2.19
Bonds (hedging, normal)	-5.54	-7.78	-9.74	-0.84	-7.78	-30.34
Bonds (hedging, oil)	0.00	0.00	0.00	0.00	0.00	-1.51
Cash	-8.93	-10.44	-11.73	-5.52	-10.44	-21.18

Table 5: Optimal asset allocation for varying levels of risk aversion (γ), rate of time preference (δ) and oil income variance ($\sigma^2_{(o)}$). We vary the level of $\sigma^2_{(o)}$ by scaling it by the factors reported in the table header: 0.85, 1.00 and 1.15, where the middle column corresponds to the in-sample estimate of $\sigma^2_{(o)}$. The sub-categories in italics refer to sub-components of total demand which relate to the normal hedging and oil hedging components as discussed in Section (3.3).

With regard to the ratio of equity to bonds in the optimal portfolio, we find that for $\gamma = 6$ and $\gamma = 9$, it ranges between 1.76 and 3.62, which is comparable to the ratios seen in the largest SWFs; see Figure 1. This holds for different levels of δ and $\sigma^2_{(o)}$. However, for very low or very high levels of risk aversion, we see optimal equity-to-bond ratios that are either too low or too high compared to observed actuals at existing funds.

We now turn to discussing the impact of changes in the correlation structure between shocks to oil income and the innovations to the financial market variables by changing β. This not only changes the loadings of ξ_{t+1} on the innovations in ν_{t+1}, but it also changes the proportion of the variance in oil income that is explained by the financial shocks.[15] The results are presented in Table 6 where we set the value of δ equal to 0.95. We assume three different cases for the correlation structure, where the in-sample estimate of β (given in the first column) is used as a benchmark. First, we assume that β is a multiple of the in-sample estimate such that we use $2^*\beta$ and $2.7^*\beta$ for cases 1 and 2, respectively. For case 3, we only change β_2 and β_4, which correspond to the loadings on the innovations to excess equity returns and the dividend yield, respectively. Specifically, for case 3 we assume $\beta_2 = 3.10$ and $\beta_4 = 40.73$, while β_1 and β_3 remain unchanged.

For case 1, doubling the loadings on the innovations to the financial market variables lowers the proportion of the variance of oil income shocks that is explained by the oil idiosyncratic shock from 0.83 (based on in-sample estimates) to about 0.53. This means that oil income is now more responsive to the financial market shocks. For low levels of risk aversion, the results do not show marked differences across the three cases. However as γ increases, the investor allocates less to equity and more to bonds, resulting in a decline in the optimal equity-to-bond ratio. In case 2, in which β is scaled by a larger factor, this tendency is further reinforced. For a risk-averse investor, this is an optimal response given a financial environment which highly correlated with oil income. This can be understood as a relative flight to safety, which is offered by bonds. Interestingly for case 3, in which β_2 and β_4 are not only larger in magnitude but also have positive signs, the optimal asset allocation is roughly similar to that of case 1. This could be due to the fact that the relative proportion of oil income shocks that is driven by financial shocks is equal in both cases.

In Figure 2, we present the historical allocation obtained using (15) by plotting the data-driven optimal historical allocation between the two classes of risky assets: equity and bonds. To avoid excessive fluctuations driven by the fluctuations in z_t, we use a five-year end-of-period moving average. Significant changes in the optimal historical allocation tracks surprisingly well

---

[15]Note that the variance of ξ_{t+1} is decomposed as $\beta^{'} \Sigma_\nu \beta + \sigma^2_{(o)}$.

	Correlation structure			
	In-sample	Case 1	Case 2	Case 3
$\gamma = 3$				
Equity (total demand)	132.67	131.74	130.64	131.70
Equity (hedging, normal)	30.64	29.71	28.60	29.71
Equity (hedging, oil)	0.01	0.02	0.02	-0.03
Bonds (total demand)	96.66	97.20	97.84	97.21
Bonds (hedging, normal)	-9.31	-8.77	-8.13	-8.77
Bonds (hedging, oil)	0.01	0.01	0.01	0.02
Cash	-129.33	-128.94	-128.47	-128.91
$\gamma = 6$				
Equity (total demand)	91.63	88.52	85.06	88.45
Equity (hedging, normal)	40.58	37.47	34.01	37.47
Equity (hedging, oil)	0.01	0.02	0.02	-0.06
Bonds (total demand)	49.35	51.01	52.80	51.07
Bonds (hedging, normal)	-3.67	-2.01	-0.22	-2.00
Bonds (hedging, oil)	0.00	0.01	0.01	0.05
Cash	-40.98	-39.52	-37.86	-39.51
$\gamma = 9$				
Equity (total demand)	84.65	77.77	70.94	77.68
Equity (hedging, normal)	50.60	43.71	36.88	43.72
Equity (hedging, oil)	0.01	0.03	0.02	-0.08
Bonds (total demand)	32.57	36.32	39.83	36.41
Bonds (hedging, normal)	-2.80	0.95	4.47	0.98
Bonds (hedging, oil)	0.00	0.01	0.01	0.06
Cash	-17.22	-14.09	-10.77	-14.09
$\gamma = 12$				
Equity (total demand)	91.67	76.80	64.99	76.54
Equity (hedging, normal)	66.09	51.23	39.42	51.25
Equity (hedging, oil)	0.04	0.03	0.03	-0.28
Bonds (total demand)	18.77	27.40	33.67	27.74
Bonds (hedging, normal)	-7.78	0.86	7.13	0.94
Bonds (hedging, oil)	0.00	0.00	0.00	0.23
Cash	-10.44	-4.20	1.34	-4.28
Variance proportion explained by $\sigma^2_{(o)}$	0.84	0.53	0.14	0.53

Table 6: Optimal asset allocation for different correlation structures between oil income shocks and innovations to the financial market variables. The first column "In-sample" refers to the in-sample estimates, while cases 1 and 2 refer to different correlation structures in which the parameters in β are multiples of the in-sample estimates. Specifically, we use $2^*\beta$ and $2.7^*\beta$ for cases 1 and 2, respectively. For case 3, we only change β_2 and β_4, such that $\beta_2 = 3.10$ and $\beta_4 = 40.73$, while β_1 and β_3 remain unchanged. The sub-categories in italics refer to sub-components of total demand which relate to the normal hedging and oil hedging components as discussed in Section (3.3).
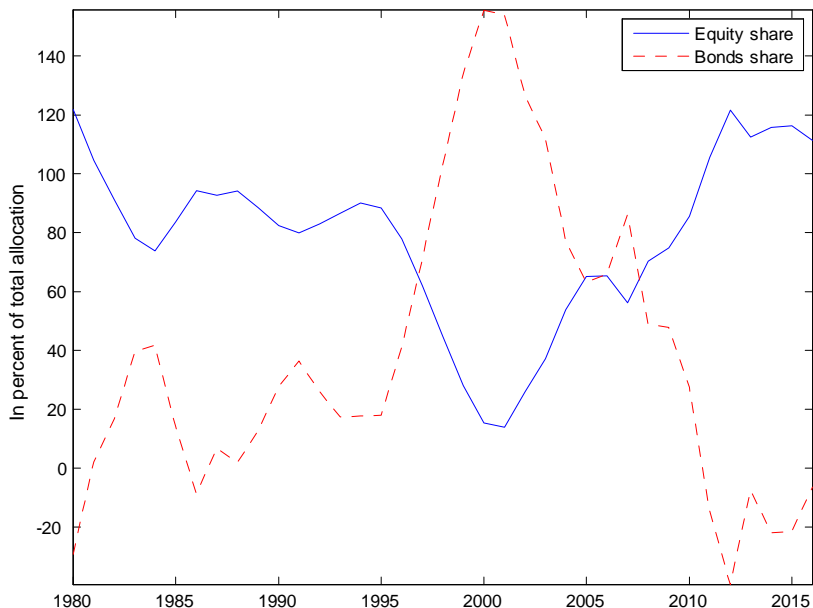
Figure 2: Optimal historical allocation to equity and bonds over the sample period 1980-2016. We use a five-year end-of-period moving average. The shares are derived using the optimal rule in (15) and the historical z_t series.

the equity boom and bust cycles over the sample period. For instance, note the gradual decline in the optimal allocation to equity prior to the bursting of the dotcom bubble in 2000. Between 1995 and 1999, excess equity returns averaged around 23.5 percent annually. With overpriced stocks and declining dividend yields, the optimal response was to change the equity-bond mix in favor of bonds. Also, note that the VAR model in (4) implies mean reversion to the unconditional mean given by $\left(I - \widehat{\Phi}_1 \right)^{-1} \widehat{\Phi}_0 = \begin{pmatrix} 0.024 & 0.080 & 0.034 & 0.021 \end{pmatrix}'$, in which the steady-state excess rate of return on equity is only 8 percent, thus strong mean reversion was bound to happen.

After the 2000 crash in equity markets, the optimal response was to rebalance the portfolio in favor of equity. Similar behavior is evident during the financial crisis of 2008, and the optimal allocation since then points to a gradual change in favor of equity. Some of the world's largest funds have actually changed their equity-bond mix during that period by incrementally increasing the share of equity in their allocations, as in the case of the Norwegian Government Pension Fund (Global) and also some of the SWFs in the Middle East. Towards the end of the sample period, this trend is partially reversed given the rally in equity markets in the last few years, and an expectation of an imminent correction implied by mean reversion.

From (16), we can see that optimal consumption depends on the path of f_t, y_t and z_t. In the evolution of c_t, b plays a crucial role as it represents the marginal propensity to consume out of

wealth. It also determines the weight on wealth f_t, relative to the weight on the stream of oil income y_t. The value of b is determined by the optimal (mean) ratios of wealth to income and consumption to income, and as discussed in Section 3.5.1, the value of b tends to be close to 1 when dealing with a mature fund as in the case of Norway, for instance.

In Table 7, we can see the impact of varying γ, δ and $\sigma_{(o)}^2$ on the optimal (mean) consumption-to-wealth ratio $\mathrm{E}\,[C_t/F_t]$.[16] We first note that for a fixed γ, the optimal consumption-to-wealth ratio declines uniformly except when γ is very high ($\gamma = 12$) and for a high level of oil income volatility; see the last column of the bottom panel.[17] This behavior is expected as an increase in δ indicates a larger weight on the utility of future consumption which leads to lower consumption out of wealth to speed up the wealth accumulation process. Similar behavior is evident when $\sigma_{(o)}^2$ gets larger. Higher variability in the stream of oil income leads the investor to scale down her consumption out of wealth to achieve a smoother consumption path.

With regard to γ, and as the level of risk aversion increases, there is also a general decline in consumption out of wealth. This occurs for different levels of δ and $\sigma_{(o)}^2$, with the exception indicated above. In the majority of the reported cases, the optimal value of b is close to 1. This means the investor behaves rather similarly to an investor at retirement since all consumption is financed by wealth, and the elasticity of consumption with respect to income $(1-b)$ becomes close to zero; see Veceira (2001) for a related discussion. This forces the investor to follow a conservative investment strategy as discussed earlier, since she tends to reduce the equity-to-bond ratio in her allocation. On the other hand, we find the optimal value of b to be around 0.79 when $\sigma_{(o)}^2$ is scaled up by 15 percent and given a large γ. This case resembles the behavior of an investor that is still a bit further from retirement, and thus her annual stream of income still plays a role in determining her consumption. This investor can afford to take on more risk and allocates a relatively higher share to equity. In addition, this investor's consumption out of wealth, $\mathrm{E}\,[C_t/F_t]$, would be lower since her consumption still partly depends on the stochastic stream of income.

It is interesting to note that the observed behavior of a SWF such as that of Norway is closest to the results in the panels corresponding to $\gamma = 9$ and $\gamma = 12$, with oil income volatility as reported in the middle column, or the right column. Since 2001, the fiscal rule in Norway has been to withdraw around 4 percent out of the fund, and it is expected to be reduced to 3 percent

---

[16]Note that the model solution gives the optimal values for the logarithm of C_t and F_t. When obtaining the expectation $\mathrm{E}\,[C_t/F_t]$, we ignore Jensen's inequality term.

[17]We note that the value of the optimal b is smaller for this parameter configuration as its value is about 0.79, which explains the marginal rise in the optimal consumption out of wealth compared to a lower value of γ.

		$\sigma^2_{(o)}$		
		0.85	1.00	1.15
$\gamma = 3$				
	0.90	0.136	0.132	0.129
δ	0.95	0.121	0.118	0.114
	0.99	0.110	0.106	0.102
$\gamma = 6$				
	0.90	0.100	0.094	0.086
δ	0.95	0.093	0.086	0.078
	0.99	0.087	0.080	0.072
$\gamma = 9$				
	0.90	0.077	0.066	0.054
δ	0.95	0.072	0.061	0.048
	0.99	0.068	0.057	0.044
$\gamma = 12$				
	0.90	0.057	0.041	0.029
δ	0.95	0.053	0.036	0.036
	0.99	0.050	0.032	0.045

Table 7: Optimal consumption-to-wealth ratio for different levels of risk aversion (γ), rate of time preference (δ) and oil income variance ($\sigma^2_{(o)}$). We vary the level of $\sigma^2_{(o)}$ by scaling it by the factors reported in the table header: 0.85, 1.00 and 1.15, where the middle column corresponds to the in-sample estimate of $\sigma^2_{(o)}$.

in the near future given the decline in the annual rate of return on the fund.

Finally, in Figure 3, we present projections of the optimal asset allocation until 2030. The objective is to show how fast the model projections converge to the steady state. For each year starting 2017, and letting s denote the forecast horizon, the optimal allocation is obtained using

$$z_{t+s} = \widehat{\Phi}_0 + \widehat{\Phi}_1 z_{t+s-1},$$

where $\widehat{\Phi}_0$ and $\widehat{\Phi}_1$ are the in-sample estimates. The results show that for different levels of γ, there should be a change in the equity-bond mix in favor of bonds. The extent of rebalancing required depends of course on γ. For the more moderate values of γ (e.g. $\gamma = 6$ and $\gamma = 9$), we see convergence to equity and bonds shares that imply an equity-to-bond ratio in the range of 1.36 to 1.88. These optimal ratios are somewhat lower than what is seen today in some of the largest SWFs. For instance, in Norway and Saudi Arabia, the ratio is around 1.95, while for other funds it often exceeds 3.[18] It would be interesting to observe how this ratio evolves at

---

[18]It should be noted, however, that some of these funds allocate a significant share to "other" asset classes, while we restrict our analysis to only equity and bonds for expositional convenience.
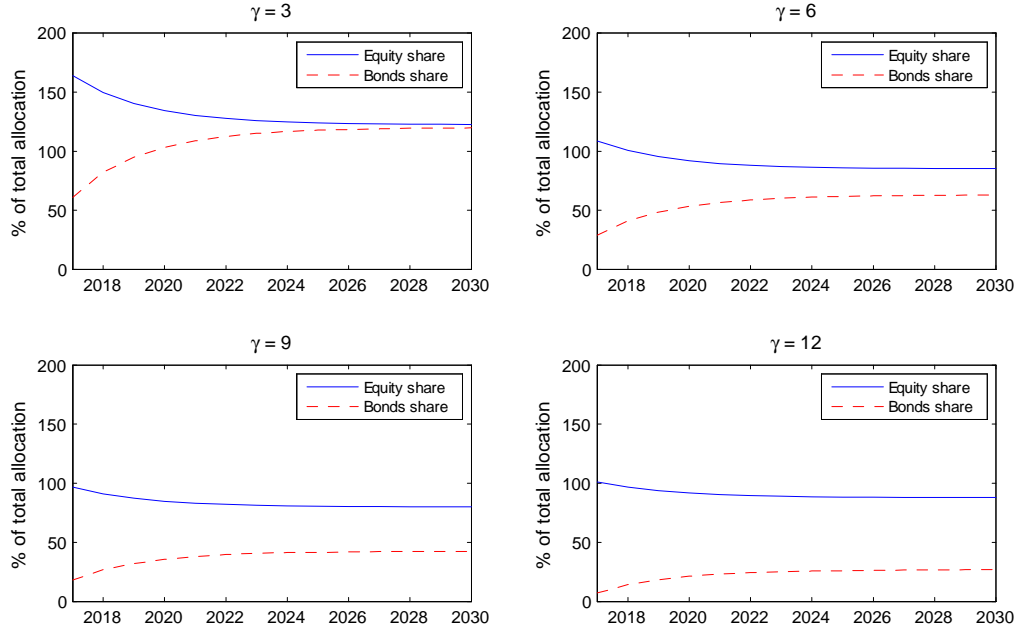
Figure 3: Projections of the optimal asset allocation for a mature fund until 2030. The assumed value of δ is equal to 0.95, and using the in-sample estimate of $\sigma^2_{(o)}$.

large SWFs over the coming years, as both the equity and bond markets revert to their historical mean rates of return.

7   Conclusion

This paper considers the problem of optimal asset allocation for a SWF subject to having a stochastic stream of oil income in the intertemporal budget constraint. Using CRRA utility, we assume the fund's objective is to maximize the discounted utility of intertemporal consumption, and use a log-linear approximation to solve for the model's optimal asset allocation and optimal consumption path. We calibrate the model using parameter estimates based on historical data for U.S. equity and bonds as two broad asset classes, in addition to data on the annual net cash flow from oil for the Norwegian state.

Using the calibrated model, we show how the optimal asset allocation and path for consumption change with variation in the model's main parameters, namely risk aversion, rate of time preference and the variance of oil income shocks. We also study how the optimal allocation changes with the correlation structure between oil income shocks and the innovations to the excess returns on risky assets. The model's predictions are consistent with predictions from

comparable models that studied optimal asset allocation subject to receiving stochastic labor income. They are also consistent to a large extent with what is observed in existing funds, especially those that are mature in the sense that they have accumulated a significant wealth buffer relative to the annual stream of oil income, as in, for example, the Government Pension Fund (Global) of Norway.

An interesting aspect of the SWF optimal asset allocation problem is the fact that the stochastic stream of income is partially under the optimizer's control, that is the rate of oil extraction can also be considered a control variable in the optimization problem. Allowing the rate of extraction to be an additional control variable is considered in Scherer (2011) and van den Bremer et al. (2016), and in the latter study the problem is largely viewed as the optimal rate of transformation of under- to above-ground wealth. This relates to the classical work of Hotelling (1931), and also subsequent contributions by Pindyck (1978, 1981), among others. Our model does not address this issue, and it remains an interesting avenue for future research to generalize our model to incorporate the rate of optimal extraction as an additional control variable.

References

Balding, C. and Y. Yao (2011). Portfolio allocation for sovereign wealth funds in the shadow of commodity-based national wealth. *International Finance Review 12*, 293–312.

Bernstein, S., J. Lerner, and A. Schoar (2013). The investment strategies of sovereign wealth funds. *Journal of Economic Perspectives 27*, 219–238.

Bodie, Z., R. C. Merton, and W. F. Samuelson (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control 16*, 427–449.

Campbell, J. Y. (1993). Intertemporal asset pricing without consumption data. *American Economic Review 83*, 487–512.

Campbell, J. Y., Y. L. Chan, and L. M. Viceira (2003). A multivariate model of strategic asset allocation. *Journal of Financial Economics 67*, 41–80.

Campbell, J. Y. and R. J. Shiller (1988). The dividend-price ratio and expectations of future dividends and discount factors. *Review of Financial Studies 1*, 195–228.

Campbell, J. Y. and L. M. Viceira (1999). Consumption and portfolio decisions when expected returns are time varying. *Quarterly Journal of Economics 114*, 433–495.

Dyck, A. and A. Morse (2011). Sovereign wealth fund portfolios. Chicago Booth Research Paper no. 11-15.

Fama, E. and K. French (1988). Dividend yields and expected stock returns. *Journal of Financial Economics 22*, 3–27.

Fama, E. and G. W. Schwert (1977). Human capital and capital market equilibrium. *Journal of Financial Economics 5*, 115–146.

Gintschel, A. and B. Scherer (2008). Optimal asset allocation for sovereign wealth funds. *Journal of Asset Management 9*, 215–238.

Glosten, L. R., R. Jagannathan, and D. Runkle (1993). Relationship between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance 48*, 1779–1802.

Harvey, C. (1989). Time-varying conditional covariances in tests of asset pricing models. *Journal of Financial Economics 22*, 305–334.

Harvey, C. (1991). The world price of covariance risk. *Journal of Finance 46*, 111–157.

Hotelling, H. (1931). The economics of exhaustible resources. *Journal of Political Economy 39*, 137–175.

Koo, H. K. (1998). Consumption and portfolio selection with labor income: a continuous-time approach. *Mathematical Finance 8*, 49–65.

Lintner, J. (1965a). Security prices, risk and maximal gains from diversification. *Journal of Finance 20*, 587–615.

Lintner, J. (1965b). The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics 47*, 13–37.

Markowitz, H. (1959). *Portfolio Selection: Efficient Diversification of Investments*. New York: Wiley.

Mayers, D. (1972). Nonmarketable assets and capital market equilibrium under uncertainty. In M. C. Jensen (Ed.), *Studies in the Theory of Capital Markets*, pp. 223–248. Praeger, New York.

Merton, R. C. (1969). Lifetime portfolio selection under uncertainty: the continuous time case. *Review of Economics and Statistics 51*, 247–257.

Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory 3*, 373–413.

Newey, W. K. and D. McFadden (1994). Large sample estimation and hypothesis testing. In R. F. Engle and D. McFadden (Eds.), *The Handbook of Econometrics, Volume 4*, pp. 2111–2245. North-Holland.

Pindyck, R. S. (1978). The optimal exploration and production of nonrenewable resources. *Journal of Political Economy 86*, 841–861.

Pindyck, R. S. (1981). The optimal production of an exhaustible resource when price is exogenous and stochastic. *Scandinavian Journal of Statistics 83*, 277–288.

Scherer, B. (2009). A note on portfolio choice for sovereign wealth funds. *Financial Markets and Portfolio Management 23*, 315–327.

Scherer, B. (2011). Portfolio choice for oil-based sovereign wealth funds. *Journal of Alternative Investments 13*, 24–34.

Song, P. X. K., Y. Fan, and J. D. Kalbfleisch (2005). Maximization by parts in likelihood inference. *Journal of the American Statistical Association 100*, 1145–1169.

van den Bremer, T., F. van der Ploeg, and S. Wills (2016). The elephant in the ground: managing oil and sovereign wealth. *European Economic Review 82*, 113–131.

Veceira, L. M. (2001). Optimal portfolio choice for long-horizon investors with nontradable labor income. *Journal of Finance 56*, 433–470.

A   Appendix A: Proof of Proposition 1

In order to find the expressions for the optimal portfolio, we use the log-Euler equation. Subtracting the log Euler equation with $i = 0$ from the log Euler equation, we obtain

$$
\begin{aligned}
\mathrm{E}_t[r_{i,t+1} - r_{0,t+1}] + \frac{1}{2}V_t[r_{i,t+1} - r_{0,t+1}] \; = \;\; & Cov_t\left(\gamma[c_{t+1} - c_t], r_{i,t+1}\right) \\
& - \;\; Cov_t\left(\gamma[c_{t+1} - c_t], r_{0,t+1}\right) \\
& - \;\; \frac{1}{2}\left[Var_t(r_{i,t+1}) - Var_t(r_{0,t+1}) - Var_t(r_{i,t+1} - r_{0,t+1})\right].
\end{aligned}
$$

First note that

$$
\frac{1}{2}\left(Var_t(r_{i,t+1}) - Var_t(r_{0,t+1}) - Var_t(r_{i,t+1} - r_{0,t+1})\right) = Cov_t(r_{i,t+1}, r_{0,t+1}) - Var_t(r_{0,t+1}).
$$

Using the trivial identity

$$
c_{t+1} - c_t = c_{t+1} - y_{t+1} + y_{t+1} - y_t + y_t - c_t
$$

and the conjuncture

$$
c_{t+1} - y_{t+1} = a + b(f_{t+1} - y_{t+1}) + B_1' z_{t+1} + z_{t+1}' B_2 z_{t+1},
$$

we find

$$
\begin{aligned}
Cov_t\left(c_{t+1} - c_t, r_{i,t+1}\right) \; = \;\; & Cov_t\left(c_{t+1} - y_{t+1} + y_{t+1} - y_t, r_{i,t+1}\right) \\
= \;\; & Cov_t\left(c_{t+1} - y_{t+1}, r_{i,t+1}\right) + Cov_t\left(y_{t+1} - y_t, r_{i,t+1}\right) \\
= \;\; & Cov_t\left(b(f_{t+1} - y_{t+1}) + B_1' z_{t+1} + z_{t+1}' B_2 z_{t+1}, r_{i,t+1}\right) + Cov_t\left(y_{t+1} - y_t, r_{i,t+1}\right).
\end{aligned}
$$

Combining the last equation with the budget equation, we find

$$
\begin{aligned}
Cov_t\left(c_{t+1} - c_t, r_{i,t+1}\right) \; = \;\; & Cov_t\left(B_1' z_{t+1} + z_{t+1}' B_2 z_{t+1}, r_{i,t+1}\right) \\
& + (1 - b)Cov_t\left(y_{t+1} - y_t, r_{i,t+1}\right) + bCov_t\left(r_{F,t+1}, r_{i,t+1}\right).
\end{aligned}
$$

We define

$$
\sigma_{F,t} = [Cov_t\left(r_{F,t+1}, r_{i,t+1}\right)]_{i=1,\dots,n},
$$

$$
\sigma_{F,0,t} = Cov_t\left(r_{F,t+1}, r_{0,t+1}\right),
$$

$$
\sigma_{0,0,t} = V_t\left(r_{0,t+1}\right),
$$

$$
\sigma_{0,t} = [Cov_t\left(r_{0,t+1}, r_{i,t+1}\right)]_{i=1,\dots,n},
$$

31

$$\sigma_{Y,t} = [Cov_t\,(y_{t+1} - y_t, r_{i,t+1})]_{i=1,\ldots,n}\,,$$

and

$$\sigma_{Y,0,t} = Cov_t\,(y_{t+1} - y_t, r_{0,t+1})\,.$$

We have

$$\sigma_{F,t} - \sigma_{F,0,t} = \Sigma_{xx}\pi_t + \sigma_{0x},$$

$$\sigma_{0,t} - \sigma_{0,0,t} = \sigma_{0x},$$

and

$$\sigma_{Y,t} - \sigma_{Y,0,t} = H_x\Sigma_\nu\beta,$$

where H_x is a selection matrix that selects the vector of excess returns from the state vector. Following Campbell et al. (2003), we find

$$Cov_t\left(B_1'z_{t+1} + z_{t+1}'B_2z_{t+1}, r_{i,t+1} - r_{0,t+1}\right) = \left(\Sigma_\nu^{(i)}\right)'B_1 + \left(\Sigma_\nu^{(i)}\right)'(B_2 + B_2')(\Phi_0 + \Phi_1 z_t),$$

where $\Sigma_\nu^{(i)}$ denotes the i-th column of Σ_ν. We define

$$M = \left[Cov_t\left(B_1'z_{t+1} + z_{t+1}'B_2z_{t+1}, r_{i,t+1} - r_{0,t+1}\right)\right]_{i=1,\ldots,n}$$

We can state that

$$M = \Lambda_0 + \Lambda_1 z_t,$$

with

$$\Lambda_0 = H_x\Sigma_\nu'\left(B_1 + (B_2 + B_2')\Phi_0\right)$$

and

$$\Lambda_1 = H_x\Sigma_\nu'\left(B_2 + B_2'\right)\Phi_1.$$

Using the matrices defined above, the log-Euler equation could be written as

$$
\begin{aligned}
H_x\Phi_0 + H_x\Phi_1 z_t + \frac{1}{2}\sigma_x^2 &= \gamma(1 - b)H_x\beta'\Sigma_\nu + \gamma M \\
&\quad + b\gamma(\sigma_{F,t} - \sigma_{F,0,t}) - (\sigma_{0,t} - \sigma_{0,0,t}),
\end{aligned}
$$

where the H_x is a selection matrix that selects the vector of excess returns from the full state vector. It follows that

$$
\begin{aligned}
H_x\Phi_0 + H_x\Phi_1 z_t + \frac{1}{2}\sigma_x^2 &= \gamma(1 - b)H_x\beta'\Sigma_\nu + \gamma(\Lambda_0 + \Lambda_1 z_t) \\
&\quad + b\gamma(\Sigma_{xx}\pi_t + \sigma_{0x}) - \sigma_{0x}.
\end{aligned}
$$

Then, we obtain

$$\pi_t = A_0 + A_1 z_t,$$

with

$$A_0 = \frac{1}{b\gamma}\Sigma_{xx}^{-1}\left[H_x\Phi_0 + \frac{1}{2}\sigma_x^2 + \sigma_{0x} - \gamma(1-b)H_x\beta'\Sigma_\nu - \gamma\Lambda_0\right] - \sigma_{0x},$$

and

$$A_1 = \frac{1}{b\gamma}\Sigma_{xx}^{-1}\left[H_x\Phi_1 - \gamma\Lambda_1\right].$$

Now, we need to show that optimal consumption is a quadratic function of the state variables. Using the Euler equation for $i = F$, we find

$$\mathrm{E}_t\left[c_{t+1} - c_t\right] = \frac{log(\delta)}{\gamma} + \chi_{F,t} + \frac{\mathrm{E}_t\left[r_{F,t+1}\right]}{\gamma}, \tag{A.1}$$

where

$$\chi_{F,t} = \frac{\gamma}{2}Var_t\left[c_{t+1} - c_t - \frac{r_{F,t+1}}{\gamma}\right].$$

Using the equation Eqn.A.1 combined with

$$c_{t+1} - c_t = c_{t+1} - y_{t+1} + y_{t+1} - y_t + y_t - c_t,$$

$$c_{t+1} - y_{t+1} = a + b(f_{t+1} - y_{t+1}) + B_1'z_{t+1} + z_{t+1}'B_2 z_{t+1} + g + \beta'\nu_{t+1} + \sigma_{(o)}\xi_{t+1}^{(o)}$$

and the budget equation, we obtain

$$\begin{aligned}
c_t - y_t &= \frac{1}{1+b\rho_c}\left(a + bk + (1-b)g - \frac{log(\delta)}{\gamma} + b\rho_f(f_t - y_t)\right)\\
&+ \frac{1}{1+b\rho_c}\left[\left(b - \frac{1}{\gamma}\right)\mathrm{E}_t\left[r_{F,t+1}\right] - \chi_{F,t} + \mathrm{E}_t\left[B_1'z_{t+1} + z_{t+1}'B_2 z_{t+1}\right]\right].
\end{aligned}$$

The conditional expectations are calculated as follows. First note that

$$\begin{aligned}
B_1'z_{t+1} &= B_1'(\Phi_0 + \Phi_1 z_t + \nu_{t+1})\\
&= B_1'\Phi_0 + B_1'\Phi_1 z_t + B_1'\nu_{t+1},
\end{aligned}$$

and

$$\begin{aligned}
z_{t+1}'B_2 z_{t+1} &= (\Phi_0 + \Phi_1 z_t + \nu_{t+1})'B_2(\Phi_0 + \Phi_1 z_t + \nu_{t+1})\\
&= \Phi_0'B_2\Phi_0 + \Phi_0'B_2\Phi_1 z_t + \Phi_0'B_2\nu_{t+1}\\
&\quad z_t'\Phi_1'B_2\Phi_0 + z_t'\Phi_1'B_2\Phi_1 z_t + z_t'\Phi_1'B_2\nu_{t+1}\\
&\quad \nu_{t+1}'B_2\Phi_0 + \nu_{t+1}'B_2\Phi_1 z_t + \nu_{t+1}'B_2\nu_{t+1}.
\end{aligned}$$

Recall that

$$\nu_{t+1} \sim N\left(0, \Sigma_\nu\right),$$

It implies

$$\mathrm{E}_t\left[B_1' z_{t+1}\right] = B_1' \Phi_0 + B_1' \Phi_1 z_t,$$

and

$$\mathrm{E}_t\left[z_{t+1}' B_2 z_{t+1}\right] = \Phi_0' B_2 \Phi_0 + \Phi_0'(B_2 + B_2')\Phi_1 z_t + z_t' \Phi_1' B_2 \Phi_1 z_t + vec(B_2)' vec(\Sigma_\nu).$$

Following Campbell et al. (2003), one can write

$$\mathrm{E}_t\left[r_{F,t+1}\right] = \Gamma_0 + \Gamma_1 z_t + z_t' \Gamma_2 z_t,$$

with

$$\Gamma_0 = A_0' H_x \Phi_0 + H_1 \Phi_0 + \frac{1}{2} A_0' \sigma_x^2 - \frac{1}{2} A_0' \Sigma_{xx} A_0,$$

$$\Gamma_1 = \Phi_0' H_x' A_1 + A_0' H_x \Phi_1 + H_1 \Phi_1 + \frac{1}{2} \sigma_x^2 A_1' - A_0' \Sigma_{xx} A_1$$

and

$$\Gamma_2 = A_1' H_x \Phi_1 - \frac{1}{2}\left(A_1' \Sigma_{xx} A_1\right).$$

We can also show that $\chi_{F,t}$ has a quadratic expression in terms of z_t. We have

$$
\begin{aligned}
c_{t+1} - c_t - \frac{r_{F,t+1}}{\gamma} &= (1-b)(y_{t+1} - y_t) + \left(b - \frac{1}{\gamma}\right) r_{F,t+1} \\
&\quad + B_1' z_{t+1} + z_{t+1}' B_2' z_{t+1},
\end{aligned}
$$

where

$$y_{t+1} - y_t = \beta' \nu_{t+1} + \sigma_{(o)} \xi_{t+1}^{(o)} + g,$$

$$r_{F,t+1} = \left[A_0' H_x + H_1\right] \nu_{t+1} + z_t' A_1 H_x \nu_{t+1} + \text{t terms and constants}.$$

and

$$
\begin{aligned}
B_1' z_{t+1} + z_{t+1}' B_2' z_{t+1} &= \left[B_1' + \Phi_0'(B_2 + B_2')\right] \nu_{t+1} + z_t' \Phi_1'(B_2 + B_2')\nu_{t+1} \\
&\quad + \nu_{t+1}' B_2 \nu_{t+1} + \text{t terms and constants}.
\end{aligned}
$$

Ignoring constants and t terms, one can write the argument of the variance $Var_t\left[c_{t+1} - c_t - \frac{r_{F,t+1}}{\gamma}\right]$ as follows

$$c_{t+1} - c_t - \frac{r_{F,t+1}}{\gamma} = \left(\Omega_1 + z_t' \Omega_2\right) \nu_{t+1} + \nu_{t+1}' B_2 \nu_{t+1} + \sigma_{(o)} \xi_{t+1}^{(o)},$$

where

$$\Omega_1 = (1-b)\beta' + \left(b - \frac{1}{\gamma}\right)\left[A_0'H_x + H_1\right] + \left[B_1' + \Phi_0'(B_2 + B_2')\right],$$

and

$$\Omega_2 = \left(b - \frac{1}{\gamma}\right)A_1 H_x + \Phi_1'(B_2 + B_2').$$

Note that Ω_1 could be written as

$$\Omega_1 = \Omega_1^{(y)} + \Omega_1^{(z)},$$

where

$$\Omega_1^{(y)} = (1-b)\beta',$$

and

$$\Omega_1^{(z)} = \left(b - \frac{1}{\gamma}\right)\left[A_0'H_x + H_1\right] + \left[B_1' + \Phi_0'(B_2 + B_2')\right].$$

Thus,

$$\chi_{F,t} = V_0 + V_1 z_t + z_t V_2 z_t',$$

with

$$V_0 = \frac{\gamma}{2}\left[\sigma_{(o)}^2 + \Omega_1 \Sigma_\nu \Omega_1' + vec\,(B_2)' Var_t\left[vec\left(\nu_{t+1}\nu_{t+1}'\right)\right] vec\,(B_2)\right],$$

$$V_1 = \gamma\left[\Omega_1 \Sigma_\nu \Omega_2'\right],$$

and

$$V_2 = \frac{\gamma}{2}\left(\Omega_2 \Sigma_\nu \Omega_2'\right).$$

The expression for V_1 could be given as

$$V_1 = V_1^{(y)} + V_1^{(z)},$$

with

$$V_1^{(y)} = \gamma\left[\Omega_1^{(y)} \Sigma_\nu \Omega_2'\right],$$

and

$$V_1^{(z)} = \gamma\left[\Omega_1^{(z)} \Sigma_\nu \Omega_2'\right].$$

Similarly, the expression for V_0 has the following decomposition

$$V_0 = V_0^{(y)} + V_0^{(z)},$$

where

$$V_0^{(y)} = \frac{\gamma}{2}\left[\sigma_{(o)}^2 + \Omega_1^{(y)} \Sigma_\nu \Omega_1^{(y)'} + \Omega_1^{(z)} \Sigma_\nu \Omega_1^{(y)'} + \Omega_1^{(y)} \Sigma_\nu \Omega_1^{(z)'}\right],$$

and

$$V_0^{(z)} = \frac{\gamma}{2} \left[\Omega_1^{(z)} \Sigma_\nu \Omega_1^{(z)'} + vec\,(B_2)' \, Var_t \left[vec \left(\nu_{t+1} \nu_{t+1}' \right) \right] vec\,(B_2) \right].$$

Note that $\Omega_1^{(z)}$ appears in $V_0^{(y)}$ but its term is of a lower order of magnitude. This implies

$$c_t - y_t = a + b(f_t - y_t) + B_1' z_t + z_t' B_2 z_t,$$

where

$$
\begin{aligned}
a &= \frac{1}{b\rho_c} \left[bk + (1-b)g - \frac{log(\delta)}{\gamma} + \left(1 - \frac{1}{\gamma} \right) \Gamma_0 - V_0 \right] \\
&\quad + \frac{1}{b\rho_c} \left[B_1' \Phi_0 + \Phi_0' B_2 \Phi_0 + vec(B_2)' vec(\Sigma_\nu) \right],
\end{aligned}
$$

$$b = \frac{\rho_f - 1}{\rho_c},$$

$$B_1 = [\rho_f I_n - \Phi_1]^{-1} \left[\left(1 - \frac{1}{\gamma} \right) \Gamma_1 - V_1^{(y)} - V_1^{(z)} + \Phi_0'(B_2 + B_2')\Phi_1 \right]'.$$

$$B_2 = \frac{1}{\rho_f} \left[\left(1 - \frac{1}{\gamma} \right) \Gamma_2 - V_2 + \Phi_1' B_2 \Phi_1 \right].$$

The coefficient a could be written as

$$a = a^{(y)} + a^{(z)},$$

with

$$a^{(y)} = \frac{1}{\rho_f - 1} \left[(1-b)g - V_0^{(y)} \right]$$

$$
\begin{aligned}
a^{(z)} &= \frac{1}{\rho_f - 1} \left[bk - \frac{log(\delta)}{\gamma} + \left(1 - \frac{1}{\gamma} \right) \Gamma_0 - V_0^{(z)} \right] \\
&\quad + \frac{1}{\rho_f - 1} \left[B_1' \Phi_0 + \Phi_0' B_2 \Phi_0 + vec(B_2)' vec(\Sigma_\nu) \right].
\end{aligned}
$$

Also, note that B_1 is as follows

$$B_1 = B_1^{(y)} + B_1^{(z)},$$

where

$$B_1^{(y)} = -[\rho_f I_n - \Phi_1]^{-1} \left[V_1^{(y)} \right]',$$

and

$$B_1^{(z)} = [\rho_f I_n - \Phi_1]^{-1} \left[\left(1 - \frac{1}{\gamma} \right) \Gamma_1 - V_1^{(z)} + \Phi_0'(B_2 + B_2')\Phi_1 \right]'.$$

It follows that

$$\Lambda_0 = \Lambda_0^{(y)} + \Lambda_0^{(z)},$$

with

$$\Lambda_0^{(y)} = H_x \Sigma_\nu' B_1^{(y)}$$

and

$$\Lambda_0^{(z)} = H_x \Sigma_\nu' \left(B_1^{(z)} + (B_2 + B_2') \Phi_0 \right).$$

Then, we find the following decomposition of the optimal solution

$$A_0 = A_0^{(m)} + A_0^{(h)} + A_0^{(o)},$$

such that

$$A_0^{(m)} = \frac{1}{b\gamma} \Sigma_{xx}^{-1} \left[H_x \Phi_0 + \frac{1}{2} \sigma_x^2 + \sigma_{0x} \right] - \sigma_{0x},$$

$$A_0^{(h)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[\Lambda_0^{(z)} \right],$$

and

$$A_0^{(o)} = \frac{1}{b} \Sigma_{xx}^{-1} \left[-(1-b) H_x \beta' \Sigma_\nu - \Lambda_0^{(y)} \right].$$

We also have

$$A_1 = A_1^{(m)} + A_1^{(h)},$$

where

$$A_1^{(m)} = \frac{1}{b\gamma} \Sigma_{xx}^{-1} \left[H_x \Phi_1 \right],$$

and

$$A_1^{(h)} = \frac{-1}{b} \Sigma_{xx}^{-1} \Lambda_1.$$

The optimal consumption could be written as

$$c_t = a^{(y)} + a^{(z)} + b f_t + (1-b) y_t + \left(B_1^{(y)'} + B_1^{(z)'} \right) z_t + z_t' B_2 z_t,$$

with $B_1' = B_1^{(y)'} + B_1^{(z)'}$.